Journal of Statistical Physics, Vol. 77, Nos. 1/2, 1994

Stationary States and Hydrodynamics of FHP Cellular Automata

J. Fritz[1]

Received June 23, 1993; final February 4, 1994

The large-scale behavior of FHP-type cellular automata is investigated in the presence of some additional random effects. It is shown that every translation-invariant stationary state of the modified model is a superposition of product measures. By means of the entropy argument of Yau and of Olla, Varadhan, and Yau, the macroscopic (Euler-type) equations governing the hydrodynamic behavior of FHP automata are also derived.

KEY WORDS: Euler equations; hydrodynamic limit; local equilibrium; ergodicity; reversible processes; relative entropy.

1. INTRODUCTION

We consider particles sitting at the points of the two-dimensional triangular lattice $\mathbb{L}$. Each particle has some velocity $v \in \mathbb{V} \equiv \{e^\circ, e', e'', -e^\circ, -e', -e''\}$, where e°, e', e'' are distinguished unit vectors of $\mathbb{L}$ such that $e^\circ + e' + e'' = 0$, i.e., each angle between them is just 120°; thus $\mathbb{L}$ is the additive subgroup of $\mathbb{R}^2$ generated by $\mathbb{V}$. In general, if $v \in \mathbb{V}$, then v' is obtained from v by rotating it by $+120°$ and $v'' \equiv (v')'$, that is, $e' = (e^\circ)'$, $e'' = (e^\circ)''$, etc. The configuration space is restricted by an exclusion rule: particles at the same site must have different velocities; empty sites are allowed. At unit times the particles sitting at the same site undergo random collisions and pass to the next site in the direction of their velocities.[1] Since the evolution preserves the total number and momentum of particles, FHP automata can be considered as a primitive caricature of gas dynamics; the related macroscopic equations are similar to the true Euler equations. On the other hand, such models can effectively be realized on a

Dedicated to Oliver Penrose on the occasion of his 65th birthday.

[1] Mathematical Institute, Hungarian Academy of Sciences and Institut des Hautes, Etudes Scientifiques, Bures-sur-Yvette, France. E-mail: h1147fri@huella.bitnet.

0022-4715/94/1000-0053$07.00/0

computer, and fairly encouraging experiments suggest that this procedure of solving the macroscopic conservation laws is less sensitive to instabilities than the traditional numerical approximation schemes. Unfortunately the randomized collisions of FHP automata are not sufficient to ensure the ergodic behavior of the system; we need some more randomness to derive the macroscopic conservation laws.[10,4] Of course, the requirement of computational simplicity restricts very strongly the scale of random effects one can add to the dynamics to regularize it.

The configurations of our system are finite or countable subsets X of $\mathbb{X} \equiv \mathbb{L} \times \mathbb{V}$; thus the configuration space is $\mathscr{X} = 2^{\mathbb{X}}$. We equip $\mathscr{X}$ with its natural product topology and Borel field $\mathscr{F}$. There are two ways to represent a configuration $X \in \mathscr{X}$, namely $\eta = \eta^X$ and $\nu = \nu^X$, where η: $\mathbb{L} \times \mathbb{V} \mapsto \{0, 1\}$ denotes the occupation number, while ν: $\mathbb{L} \mapsto \mathscr{V} \equiv 2^{\mathbb{V}}$ is the lattice version of the very same configuration $X \subset \mathbb{X}$. More precisely, $\eta^X = \eta^X(q, v) = 1$ if we have a particle of velocity $v \in \mathbb{V}$ at site $q \in \mathbb{L}$, i.e., if $(q, v) \in X$, and it is zero otherwise. On the other hand, $\nu^X = (\nu_q^X)_{q \in \mathbb{L}}$, where ν_q^X denotes the set of velocities of the particles sitting at site $q \in \mathbb{L}$; i.e., $\nu_q^X = \{v \in \mathbb{V}: (q, v) \in X\}$.

Total number and momentum of the particles of a configuration $X \in \mathscr{X}$ at site $q \in \mathbb{L}$ are denoted by $N_q = N_q(X)$ and $P_q = P_q(X)$, respectively, where

$$N_q(X) = \sum_{v \in \mathbb{V}} \eta^X(q, v), \qquad P_q(X) = \sum_{v \in \mathbb{V}} \eta^X(q, v)\, v \tag{1.1}$$

One step of the evolution is given by a random transformation $\mathbf{T}$: $\mathscr{X} \mapsto \mathscr{X}$ of type $\mathbf{T} = \mathbf{SC}$, where $\mathbf{S}$ is the free streaming of particles and $\mathbf{C} = \mathbf{C}_\gamma$ specifies collisions governed by a random element γ.

Free Evolution. $\mathbf{S}$: $\mathscr{X} \mapsto \mathscr{X}$ is a deterministic transformation defined by $\eta^{\mathbf{S}X}(q, v) = \eta^X(q - v, v)$. It is plain that $\mathbf{S}$ does not violate the exclusion rule.

Collisions. In general, a collision rule is a one-to-one map Π: $\mathscr{V} \mapsto \mathscr{V}$ preserving the total number and momentum of particles at the given site. Nontrivial collisions are possible only if $N_q = 2, 3, 4$, but we do not use all possibilities. A subset $\mathbb{D}_v \equiv \{v, -v\}$, $v \in \mathbb{V}$, of velocities is called a dipole, and $\mathbb{T}_v \equiv \{v, v', v''\}$ is a tripole; they are the objects of collisions. The collision mechanism $\mathbf{C}_\gamma$: $\mathscr{X} \mapsto \mathscr{X}$ of a simple FHP automaton is defined by a binary sequence $\gamma = (\gamma_q)_{q \in \mathbb{L}}$, $\gamma_q = \pm 1$, as follows. For each $q \in \mathbb{L}$ we set

$$\nu_q^{\mathbf{C}_\gamma X} = \begin{cases} \mathbb{D}_{v'} & \text{if } \nu_q^X = \mathbb{D}_v \text{ and } \gamma_q = +1 \\ \mathbb{D}_{v''} & \text{if } \nu_q^X = \mathbb{D}_v \text{ and } \gamma_q = -1 \\ \mathbb{T}_{-v} & \text{if } \nu_q^X = \mathbb{T}_v \text{ and } \gamma_q = +1 \\ \nu_q^X & \text{in all other cases} \end{cases}$$

Collisions at different sites are independent of each other with common distribution $P[\gamma_q = +1] = P[\gamma_q = -1] = 1/2$ for each $q \in \mathbb{L}$. Let $\gamma^t = (\gamma_q^t)_{q \in \mathbb{L}}$, $t \in \mathbb{N}$, denote an independent sequence of sets of binary variables as specified above and let $\mathbf{C}_{\gamma^t}$ denote the corresponding collision operator; then for any initial value $X \in \mathscr{X}$ the recursion $\xi_0 = X$, $\xi_t = \mathbf{S}\mathbf{C}_{\gamma^t}\xi_{t-1}$ for $t \in \mathbb{N}$ defines a Markov chain in $(\mathscr{X}, \mathscr{F})$ called the simple FHP automaton. In case of the original model[1] the reflection of tripoles is not randomized; we have to reflect simultaneously every tripole appearing in the configuration whenever the exclusion rule allows us to do so; the associated collision operator will be denoted as $\mathbf{C}_\gamma^a$. The modification introduced above is needed to ensure the proper ergodic behavior of the system. In fact, we need even more randomness; see next section.

A distinguished family of states, the class of local equilibria, corresponds to the classical conservation laws N and P. For any set of parameters $\Gamma = (z_q, u_q)_{q \in \mathbb{L}}$, $z_q \in \mathbb{R}$ and $u_q \in \mathbb{R}^2$, we define a product measure λ_Γ on $(\mathscr{X}, \mathscr{F})$ by

$$\lambda_\Gamma[X_\Lambda] = \prod_{q \in \Lambda} \exp[z_q N_q(X) + \langle u_q, P_q(X) \rangle - F(z_q, u_q)] \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^2$, F is the normalizing term, Λ is an arbitrary finite subset of $\mathbb{L}$, and $X_\Lambda \equiv \{ Y \subset \mathbb{X}\colon\ Y \cap \Lambda \times \mathbb{V} = X \cap \Lambda \times \mathbb{V} \}$ is an elementary cylinder set in $\Lambda \subset \mathbb{L}$. The normalizing term (free energy) of the model is explicitly known; we have

$$F(z, u) = \sum_{v \in \mathbb{V}} \Psi(z + \langle u, v \rangle), \qquad \Psi(w) \equiv \log(1 + e^w) \tag{1.3}$$

for $z, w \in \mathbb{R}$ and $u \in \mathbb{R}^2$. It is easy to verify that the usual thermodynamic formalism applies, namely

$$\begin{aligned} \rho_q = \rho_q^\Gamma &\equiv \int N_q(X)\, \lambda_\Gamma(dX) = F'_z(z_q, u_q) \\ \pi_q = \pi_q^\Gamma &\equiv \int P_q(X)\, \lambda_\Gamma(dX) = F'_u(z_q, u_q) \end{aligned} \tag{1.4}$$

where F'_z and F'_u denote the corresponding partial derivatives of F; $F'_u \in \mathbb{R}^2$ is a vector. On the other hand, $z_q = S'_\rho(\rho_q, \pi_q)$ and $u_q = S'_\pi(\rho, \pi)$, where the entropy

$$S(\rho, \pi) = \sup_{z \in \mathbb{R}, u \in \mathbb{R}^2} [z\rho + \langle u, \pi \rangle - F(z, u)] \tag{1.5}$$

is finite and convex if $0 < \rho < 6$ and π belongs to the interior of the convex hull of $2\mathbb{V}$. It is perhaps unusual that ρ_q and π_q depend both on z_q and u_q

in a complicated way. However, if u_q is parallel to one of the velocities, then so is π_q, and if u_q is orthogonal to $v \in \mathbb{V}$, then π_q is also orthogonal to this velocity. The set of local equilibrium distributions λ_Γ will be denoted by $\mathscr{G}_{le}$, and $\mathscr{G}_e$ consists of the translation-invariant elements of $\mathscr{G}_{le}$. It is easy to verify that each $\lambda \in \mathscr{G}_e$ is a stationary state of the simple FHP automaton, but there are many other stationary measures of this model.

Our final purpose is to describe the macroscopic behavior of the conservative fields N and P. For $\varepsilon > 0$ and $X \in \mathscr{X}$ the density and momentum fields are defined as

$$\begin{aligned} N_\varepsilon(X, \varphi) &= \frac{\varepsilon^2 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \varphi(\varepsilon q)\, N_q(X) \\ P_\varepsilon(X, \psi) &= \frac{\varepsilon^2 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \langle \psi(\varepsilon q), P_q(X) \rangle \end{aligned} \tag{1.6}$$

where $\varphi\colon \mathbb{R}^2 \mapsto \mathbb{R}$ and $\psi\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ are continuously differentiable with compact support. The hydrodynamic law of large numbers means that under suitable initial conditions we have a deterministic limit:

$$\begin{aligned} \lim_{\varepsilon \to 0} N_\varepsilon(\xi_{[\tau/\varepsilon]}, \varphi) &= \iint \varphi(x, y)\, \rho(\tau, x, y)\, dx\, dy \\ \lim_{\varepsilon \to 0} P_\varepsilon(\xi_{[\tau/\varepsilon]}, \psi) &= \iint \langle \psi(x, y), \pi(\tau, x, y) \rangle\, dx\, dy \end{aligned} \tag{1.7}$$

in probability, where $[u]$ denotes the integer part of $u \in \mathbb{R}$, $\tilde{q} = (x, y) \approx \varepsilon q$ is the macroscopic location, and the limiting densities ρ and π are governed by a couple of partial differential equations.

Since the collisions preserve both N_q and P_q, we obtain

$$\begin{aligned} N_\varepsilon(\xi_{t+1}, \varphi) - N_\varepsilon(\xi_t, \varphi) &= \frac{\varepsilon^2 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \sum_{v \in \mathbb{V}} [\varphi(\varepsilon q + \varepsilon v) - \varphi(\varepsilon q)]\, \eta^{\xi_t}(q, v) \\ &\approx \frac{\varepsilon^3 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \sum_{v \in \mathbb{V}} \langle \nabla \varphi(\varepsilon q), v \rangle\, \eta^{\xi_t}(q, v) \\ &= \frac{\varepsilon^3 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \langle \nabla \varphi(\varepsilon q), P_q(\xi_t) \rangle \\ P_\varepsilon(\xi_{t+1}, \psi) - P_\varepsilon(\xi_t, \psi) &= \frac{\varepsilon^2 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \sum_{v \in \mathbb{V}} \langle \psi(\varepsilon q + \varepsilon v) - \psi(\varepsilon q), v \rangle\, \eta^{\xi_t}(q, v) \\ &\approx \frac{\varepsilon^3 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \sum_{v \in \mathbb{V}} \langle \nabla \psi(\varepsilon q)\, v, v \rangle\, \eta^{\xi_t}(q, v) \\ &= \frac{\varepsilon^3 \sqrt{3}}{2} \sum_{q \in \mathbb{L}} \sum_{v \in \mathbb{V}} \eta^{\xi_t}(q, v)\, \mathrm{Tr}\, \nabla \psi(\varepsilon q)\, Q_v \end{aligned} \tag{1.8}$$

where ∇ is the gradient operator, Tr denotes the usual trace in $\mathbb{R}^2$, and Q_v is the operator of orthogonal projection to $v \in \mathbb{V}$, i.e., $Q_v u = \langle u, v \rangle v$ for $u \in \mathbb{R}^2$. If e° and e^* are the unit vectors of our orthogonal system of coordinates, then the matrix of Q_v reads

$$Q_v = \begin{pmatrix} \langle e^\circ, v\rangle^2 & \langle e^\circ, v\rangle\langle e^*, v\rangle \\ \langle e^\circ, v\rangle\langle e^*, v\rangle & \langle e^*, v\rangle^2 \end{pmatrix}, \qquad \text{thus} \quad \tfrac{1}{3}\sum_{v\in\mathbb{V}} Q_v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The first equation of (1.8) turns into the continuity equation $\partial_\tau \rho + \operatorname{div} \pi = 0$ as $\varepsilon \to 0$. To evaluate the second one, we have to assume that the state of the system is close to a local equilibrium λ_Γ. Since

$$\int \eta^X(q, v)\, \lambda_\Gamma(dX) = \Psi'(z_q + \langle u_q, v\rangle); \qquad \Psi'(w) = \frac{e^w}{1+e^w} \qquad \text{for} \quad w \in \mathbb{R}$$

by a formal calculation and the divergence theorem we obtain that the macroscopic densities ρ, π satisfy

$$\partial_\tau \rho + \operatorname{div} \pi = 0, \qquad \partial_\tau \pi + \operatorname{Div} Q(\rho, \pi) = 0 \tag{1.9}$$

in a weak sense, where Q is a 2×2 symmetric matrix and Div denotes its tensorial divergence defined by the identity

$$\iint \operatorname{Tr} \nabla\psi Q\, dx\, dy + \iint \langle \psi, \operatorname{Div} Q\rangle\, dx\, dy = 0$$

Using the correspondence (1.4) between parameters and mean values of local equilibria, we get

$$Q(\rho, \pi) = \sum_{v\in\mathbb{V}} \Phi_v(\rho, \pi)\, Q_v \tag{1.10}$$

where

$$\Phi_v(\rho, \pi) \equiv \Psi'(z + \langle u, v\rangle) = \Psi'(S'_\rho(\rho, \pi) + \langle S'_\pi(\rho, \pi), v\rangle)$$

If the Euler equations (1.9) are uniquely solved, which will be assumed at least for some time interval $[0, \tau_0)$, then the evolution of the parameters is prescribed, too. Since

$$\begin{aligned} \operatorname{div} \pi &= \sum_{v\in\mathbb{V}} \Psi''(z + \langle u, v\rangle)(\langle \nabla z, v\rangle + \langle (\nabla u)\, v, v\rangle) \\ \operatorname{Div} Q &= \sum_{v\in\mathbb{V}} \Psi''(z + \langle u, v\rangle)(\langle \nabla z, v\rangle + \langle (\nabla u)\, v, v\rangle)\, v \end{aligned} \tag{1.11}$$

$\langle (\nabla u)\, v, v\rangle = \operatorname{Tr} Q_v \nabla u$, $\sum_{v\in\mathbb{V}} \Phi_v = \rho$, and $\sum_{v\in\mathbb{V}} \Phi_v v = \pi$ are identities, by a direct calculation we get an evolution for z and u in terms of their space gradients ∇z and ∇u [cf. (1.8)], namely

$$\begin{aligned} \partial_\tau z &= -\sum_{v\in\mathbb{V}} \Phi'_{v\rho}(\rho,\pi)\langle (\nabla u)\, v, v\rangle = -\operatorname{Tr}(\nabla u)\, Q'_\rho(\rho,\pi) \\ \partial_\tau u + \nabla z &= -\sum_{v\in\mathbb{V}} \Phi'_{v\pi}(\rho,\pi)\langle (\nabla u)\, v, v\rangle = -\mathbf{Tr}(\nabla u)\, Q'_\pi(\rho,\pi) \end{aligned} \tag{1.12}$$

where Φ' and Q' are the corresponding derivatives,

$$Q'_\rho(\rho,\pi) \equiv \sum_{v\in\mathbb{V}} \Phi'_{v\rho}(\rho,\pi)\, Q_v, \qquad Q'_\pi(\rho,\pi) \equiv \sum_{v\in\mathbb{V}} \Phi'_{v\pi}(\rho,\pi)\, Q_v$$

Since $\pi\in\mathbb{R}^2$, the matrix elements of Q'_π are also vectors; **Tr** is a vector obtained as the sum of the diagonal elements.

It is remarkable that the macroscopic equations admit steady-state solutions such that the velocity field is either parallel or perpendicular to one of the six velocities $v\in\mathbb{V}$. For example, let us look for solutions of type $\rho=\rho(y)$ and $\pi=(\pi_x(y),0)$, where $\pi_x \equiv \langle e^\circ, \pi\rangle$ denotes the x component of $\pi\in\mathbb{R}^2$. Then $\operatorname{div}\pi=0$ is automatically satisfied, while $\operatorname{Div} Q=0$ reduces to

$$\sum_{v\in\mathbb{V}} \Psi''(z+u_x v_x)(z'+u'_x v_x)\, v_y v = 0$$

where $z=z(y)$ and $u=(u_x(y),0)$ are the corresponding parameters, $v_x=\langle v, e^\circ\rangle$, $v_y=\langle v, e^*\rangle$ and $z'=\partial_y z$, $u'_x=\partial_y u_x$, whence by a direct calculation

$$\begin{aligned} &\Psi''\left(z-\frac{u_x}{2}\right)\left(z'-\frac{u'_x}{2}\right)+\Psi''\left(z+\frac{u_x}{2}\right)\left(z'+\frac{u'_x}{2}\right) \\ &\quad=\partial_y\left(\Psi'\left(z+\frac{u_x}{2}\right)+\Psi'\left(z-\frac{u_x}{2}\right)\right)=0 \end{aligned} \tag{1.13}$$

Consequently we have a constant $0<c<2$ such that

$$\Psi'\left(z(y)+\frac{u_x(y)}{2}\right)+\Psi'\left(z(y)-\frac{u_x(y)}{2}\right)=c \tag{1.14}$$

for all $y\in\mathbb{R}$. This equation can explicitly be solved for z given u_x,

$$\begin{aligned} z &= \log\frac{c-1}{2-c}+\log\left[\cosh\frac{u_x}{2}+\left(\cosh^2\frac{u_x}{2}+\frac{c(2-c)}{(c-1)^2}\right)^{1/2}\right] \quad \text{if} \quad 1<c<2 \\ z &= \log\frac{1-c}{2-c}+\log\left[\left(\cosh^2\frac{u_x}{2}+\frac{c(2-c)}{(c-1)^2}\right)^{1/2}-\cosh\frac{u_x}{2}\right] \quad \text{if} \quad 0<c<1 \end{aligned} \tag{1.15}$$

and $z = 0$ for any $u_x \in \mathbb{R}$ if $c = 1$. Observe that $z \leqslant z^* \equiv \log c - \log(2-c) < 0$ if $0 < c < 1$, while $z \geqslant z^* > 0$ if $1 < c < 2$.

Therefore for any differentiable $u_x : \mathbb{R} \mapsto \mathbb{R}$, (1.15) defines a stationary solution to (1.9). In the case of piecewise smooth solutions the Rankine–Hugoniot shock condition[11] reduces again to (1.14) in the sense that its left-hand side does not change across the jump, and thus we have an extremely rich variety of discontinuous steady-state solutions, too.

In view of the entropy argument of refs. 13 and 10, the derivation of the macroscopic conservation laws (1.9) reduces to the following form of the ergodic hypothesis: every translation-invariant stationary state of the evolution is a superposition of equilibrium (Gibbs) states $\lambda \in \mathscr{G}_e$ with random parameters. The same argument yields a microscopic description of smooth stationary solutions (1.15), too. In the next section we discuss some modifications of the simple FHP automata which satisfy this hypothesis.

2. MAIN RESULT

Random collisions of the simple FHP automata are not sufficient to ensure the ergodic behavior of the system; there are many other stationary states of the evolution. In particular, a product measure λ is stationary if $\lambda = \lambda_\Gamma \in \mathscr{G}_{le}$ with $z_q = z$ and $u_q = \alpha + \delta q^*$, where $\alpha \in \mathbb{R}^2$, $\delta \in \mathbb{R}$, and q^* is obtained from $q \in \mathbb{L}$ by rotating it by $+90°$. As a superposition and weak limit of such probability measures we can construct translation-invariant stationary states, but there might exist even more complex, degenerate or strongly dependent stationary states, too. The easiest way to overcome this difficulty is to introduce a random, symmetric exchange mechanism resulting in a small perturbation of the free streaming of particles. In some situations we allow a particle to jump across a bond which is not parallel to its velocity.

Random Exchanges. Let $\mathbb{L}^*$ denote the set of bonds $b = \{p, q\}$, i.e., $p, q \in \mathbb{L}$, $p - q \in \mathbb{V}$; for $u \in \mathbb{V}$ we write $u \parallel b$ if $p - q = \pm u$, $u \nparallel b$ otherwise. To define exchanges in a unique way, we have to select a pairwise disjoint set of bonds $\omega \subset \mathbb{L}^*$ somehow, and if $b \in \omega$, then we let one of the particles of velocity $v \nparallel b$ jump across b. More precisely, the exchange mechanism $\mathbf{B} \equiv \mathbf{B}_\beta : \mathscr{X} \mapsto \mathscr{X}$ is defined in terms of a set of random velocities $\beta = (\beta_q)_{q \in \mathbb{L}}$, $\beta_q \in \mathbb{V}$, as follows. Let $\omega = \omega(\beta)$ be the set of bonds $b \in \mathbb{L}^*$ for which $\beta_b \nparallel b$ and $\beta_c \parallel c$ whenever $|b \cap c| = 1$, where $|\Lambda|$ denotes the cardinality of $\Lambda \subset \mathbb{L}$; then

$$\eta^{\mathbf{B}_\beta X}(q, v) = \begin{cases} \eta^X(p, v) & \text{if } \{q, p\} = b \in \omega(\beta) \text{ and } v = \beta_b \\ \eta^X(q, v) & \text{in all other cases} \end{cases} \tag{2.1}$$

For simplicity we assume that β is a system of independently selected velocities such that for all $b \in \mathbb{L}^*$ we have $P[\beta_b = v] = \kappa$ if $v \nparallel b$, while $P[\beta_b = v] = 1/2 - 2\kappa$ if $v \parallel b$, where $0 < \kappa < 1/4$. This means that first we mark bonds independently with probability 2κ and keep only the isolated ones, then we decide independently which particle is allowed to jump. The stirring of identical particles is uniquely defined and **B** describes a diffusive perturbation of the free streaming of particles. Given an independent sequence β', $t \in \mathbb{N}$, of random systems which is completely independent of the previously introduced sequence γ', the FHP automaton with stirring is a Markov process in $\mathscr{X}$ defined by $\xi_0 = X \in \mathscr{X}$ and $\xi_t = \mathbf{S}\mathbf{B}_{\beta^t}\mathbf{C}_{\gamma^t}\xi_{t-1}$ if $t \in \mathbb{N}$. We shall see that, at least at the level of Euler scaling, this random mechanism does not change the macroscopic behavior of the system. Of course, there are many other possibilities to define such a diffusive, random perturbation of the free motion of particles.

In a stationary regime the random stirring mechanism maintains a uniform distribution of particles, all correlation functions depend only on the number of different kinds of participating particles. In view of a theorem by deFinetti such a state μ admits an ergodic decomposition into product measures: $\mu(dX) = \int \mu[dX | \mathscr{F}_{\text{inv}}]\, d\mu$, where $\mathscr{F}_{\text{inv}}$ denotes the σ-field of translation-invariant events. There are some degenerate stationary states with $v_q^X = v_0^X$ for all $q \in \mathbb{L}$; otherwise the randomized collisions imply the required symmetry of the ergodic components of the stationary state. The original collision rule $\mathbf{C}^o$ does not preserve the individual ergodic components; then the situation is little bit more complicated; see Remark 3.4. Following the ideas of ref. 4, we get:

Theorem 2.1. Every translation-invariant stationary state μ of the FHP automaton with stirring is a superposition of product measures. If $0 < \mu[\eta^X(0, v) | \mathscr{F}_{\text{inv}}] < 1$ μ-a.s., then μ is a superposition of Gibbs states $\lambda_\Gamma \in \mathscr{G}_e$ with random parameters $z \in \mathbb{R}$ and $u \in \mathbb{R}^2$, i.e., $z_q = z$ and $u_q = u$ for all $q \in \mathbb{L}$.

The above description of translation-invariant stationary states allows us to evaluate the right-hand side of (1.8); notice that the contribution of random exchanges is really negligible because

$$\eta^{\mathbf{B}_\beta X}(q, v) = \eta^X(q, v) + \sum_{y \in \mathbb{V}} \chi^v_{q,y}(\beta)[\eta^X(q + y, v) - \eta^X(q, v)] \qquad (2.2)$$

where $\chi^v_{q,y}(\beta) = 1$ if β permits a particle of velocity $v \in \mathbb{V}$ to jump across the bond $b = \{q, q + y\}$, and it is zero otherwise. Indeed, as

$$P[\chi^v_{q,y}(\beta) = 1] = \kappa(1 - 2\kappa)^{10} \qquad \text{for all} \quad q \in \mathbb{L} \quad \text{if} \quad v \nparallel y$$

and is zero otherwise, we have a diffusive perturbation of the free streaming of particles which vanishes in the Eulerian scaling limit $t \to \tau/\varepsilon$. Our calculations are based on the hypothesis that the true distribution μ_t at time $t \approx \tau/\varepsilon$ is close to a local equilibrium state $\lambda_{\Gamma(t)} \in \mathscr{G}_{le}$. In view of the ideas of refs. 13 and 10, the exact meaning of the relation $\mu_t \approx \lambda_{\Gamma(t)}$ should be given in terms of relative entropy. Here we reformulate their argument in a context of infinite volumes.

Let $\Lambda^n \subset \mathbb{L}$ denote the zero-centered hexagonal box of radius n, i.e., $\Lambda^0 = \{0\}$ while $\Lambda^n = \{q+v\colon\ q \in \Lambda^{n-1},\ v \in \mathbb{V}\}$, and define the relative entropy of two Borel probabilities μ and λ in the box Λ^n by

$$I_n[\mu \mid \lambda] = \sum_{X_{\Lambda^n}} \mu[X_{\Lambda^n}] \log \frac{\mu[X_{\Lambda^n}]}{\lambda[X_{\Lambda^n}]} \tag{2.3}$$

In view of the general philosophy we have to minimize $I_n[\mu_t \mid \lambda_{\Gamma(t)}]$ in a global sense; the optimal set of parameters is predicted by the macroscopic equations (1.9) via (1.12). The main steps of this approach can be outlined as follows. At a given level $\varepsilon > 0$ of scaling the process ξ_t is started with an ε-dependent initial distribution μ_0^ε; the distribution of ξ_t will be denoted by μ_t^ε. The argument is completed by showing that

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon t \leqslant \tau} \sup_{\varepsilon n \leqslant d} \varepsilon^2 I_n[\mu_t^\varepsilon \mid \lambda_{\Gamma^\varepsilon(t)}] = 0 \tag{2.4}$$

for all $d > 0$ and $\tau < \tau_0$, where $\Gamma^\varepsilon(t) = (z_q^\varepsilon(t), u_q^\varepsilon(t))_{q \in \mathbb{L}}$ is a suitable set of time-dependent parameters. Indeed, for any event A_n depending only on the configuration in Λ^n we have

$$\mu_t^\varepsilon[A_n] \leqslant \frac{1 + I_n[\mu_t^\varepsilon \mid \lambda_{\Gamma^\varepsilon(t)}]}{-\log \lambda_{\Gamma^\varepsilon(t)}[A_n]} \tag{2.5}$$

Thus the exponential law of large numbers we have for $\lambda_{\Gamma^\varepsilon(t)}$ implies the hydrodynamic law of large numbers. The derivation of (2.4) is based on a recursive estimation

$$I_n^\varepsilon[\mu_{t+1}^\varepsilon \mid \lambda_{\Gamma^\varepsilon(t+1)}] \leqslant (1 + \varepsilon K)\, I_{n+1}^\varepsilon[\mu_t^\varepsilon \mid \lambda_{\Gamma^\varepsilon(t)}] + R_n^\varepsilon(t) \tag{2.6}$$

where R is a remainder vanishing in the hydrodynamic limit. Since (2.6) is explicitly solved by iteration, we get (2.4) whenever the latter holds true at $\tau = 0$. Of course, the remainder shall vanish only if we choose the parameters in an optimal way as prescribed by the macroscopic equations; we use the following approximation scheme.

Given the initial values $z_q^\varepsilon(0)$ and $u_q^\varepsilon(0)$, we define $z_q^\varepsilon(t+1)$ and $u_q^\varepsilon(t+1)$ for all $q \in \mathbb{L}$ by induction. First we introduce the gradients of z and

u, ∇z_q is a vector, ∇u_q is a tensor; they are defined by their least squares estimators:

$$\nabla z_q^\varepsilon(t) \equiv \tfrac{1}{3} \sum_{v \in \mathbb{V}} [z_{q+v}^\varepsilon(t) - z_q^\varepsilon(t)]\, v \text{ and } \nabla u_q^\varepsilon(t) \equiv \tfrac{1}{3} \sum_{v \in \mathbb{V}} [u_{q+v}^\varepsilon(t) - u_q^\varepsilon(t)] \circ v \tag{2.7}$$

where $u \circ v$ denotes the diadic product of $u, v \in \mathbb{R}^2$, i.e., $u \circ v$: $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is a linear transformation such that $(u \circ v)\, y = u\langle v, y\rangle$ for all $y \in \mathbb{R}^2$; thus $v \circ v = Q_v$. It is useful to notice that we have identities above if $z_{q+v} - z_q = \langle \nabla u_q, v\rangle$ and $u_{q+v} - u_q = \nabla u_q v$.

Having in mind (1.8), (1.9), and (1.12), the evolution of parameters is defined by

$$\begin{aligned} z_q^\varepsilon(t+1) &= z_q^\varepsilon(t) - \sum_{v \in \mathbb{V}} \Phi'_{vp}(\rho_q^\varepsilon(t), \pi_q^\varepsilon(t))\langle (\nabla u_q^\varepsilon(t))\, v, v\rangle \\ u_q^\varepsilon(t+1) &= u_q^\varepsilon(t) - \nabla z_q^\varepsilon(t) - \sum_{v \in \mathbb{V}} \Phi'_{v\pi}(\rho_q^\varepsilon(t), \pi_q^\varepsilon(t))\langle (\nabla u_q^\varepsilon(t))\, v, v\rangle \end{aligned} \tag{2.8}$$

where $\rho_q^\varepsilon(t) \equiv F'_z(z_q^\varepsilon(t), u_q^\varepsilon(t))$ and $\pi_q^\varepsilon(t) \equiv F'_u(z_q^\varepsilon(t), u_q^\varepsilon(t))$.

To derive (2.6), besides ergodicity we need also some smoothness of the parameters defined by (2.8). Euler-type equations like (1.9) develop shocks in a finite time, thus we can have the necessary smoothness for a short interval $[0, \tau_0)$ of macroscopic time only; our conditions are summarized below. For each $0 < \tau < \tau_0$ and $\varepsilon > 0$ we have some finite numbers $a(\tau)$ and $b(\varepsilon, \tau)$ such that $b(\varepsilon, \tau) \to 0$ if $\tau < \tau_0$ is fixed and $\varepsilon \to 0$; moreover,

$$\begin{aligned} \varepsilon\, |z_q^\varepsilon(t)| + \varepsilon\, |u_q^\varepsilon(t)| + |\nabla z_q^\varepsilon(t)| + \|\nabla u_q^\varepsilon(t)\| &\leqslant a(\tau)\, \varepsilon \\ |z_{q+v}^\varepsilon(t) - z_q^\varepsilon(t) - \langle \nabla z_q^\varepsilon(t), v\rangle| + |u_{q+v}^\varepsilon(t) - u_q^\varepsilon(t) - \nabla u_q^\varepsilon(t)\, v| &\leqslant \varepsilon b(\varepsilon, \tau) \\ |\nabla z_{q+v}^\varepsilon(t) - \nabla z_q^\varepsilon(t)| + \|\nabla u_{q+v}^\varepsilon(t) - \nabla u_q^\varepsilon(t)\| &\leqslant \varepsilon b(\varepsilon, \tau) \end{aligned} \tag{2.9}$$

for all $q \in \mathbb{L}$, $v \in \mathbb{V}$, and $0 \leqslant \varepsilon t \leqslant \tau$, where $|u|$ denotes the length of a vector $u \in \mathbb{R}^2$, while $\|A\|$ is the usual norm of a tensor A. Let us remark that the second and third conditions above express the existence and the continuity of the gradients of the parameters.

Theorem 2.2. Suppose (2.4) for $\tau = 0$ and (2.9) for $0 < \tau < \tau_0$; then the approximation scheme (2.7) predicts the hydrodynamic behavior of the FHP automaton with exclusion for $t < \tau_0/\varepsilon$ in the sense that we have (2.4) for all $\tau < \tau_0$; consequently

$$\begin{aligned} \lim_{\varepsilon \to 0} \sum_{q \in \mathbb{L}} \varphi(\varepsilon q)(N_q(\xi_{[\tau/\varepsilon]}) - S'_z(z_q^\varepsilon([\tau/\varepsilon]), u_q^\varepsilon([\tau/\varepsilon]))) &= 0 \\ \lim_{\varepsilon \to 0} \sum_{q \in \mathbb{L}} \langle \psi(\varepsilon q), P_q(\xi_{[\tau/\varepsilon]}) - S'_\pi(z_q^\varepsilon([\tau/\varepsilon]), u_q^\varepsilon([\tau/\varepsilon]))\rangle &= 0 \end{aligned} \tag{2.10}$$

in probability for each $\tau < \tau_0$ and continuous $\varphi\colon \mathbb{R}^2 \mapsto \mathbb{R}$, $\psi\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ with compact support.

Although local equilibria corresponding to the stationary solutions (1.15) of the macroscopic equations are not stationary states of the microscopic evolution, such profiles are reproduced in the hydrodynamic limit as follows. Suppose that $z_q^\varepsilon = z_{q+e^\circ}^\varepsilon$ and $u_q^\varepsilon = u_{q+e^\circ}^\varepsilon \| e^\circ$ for all $q \in \mathbb{L}$. Then from (2.7) we get

$$\nabla z_q^\varepsilon = (z_{q+e'}^\varepsilon - z_{q+e''}^\varepsilon)\frac{e^\circ}{\sqrt{3}} \quad \text{and} \quad \langle \nabla u_q^\varepsilon v, v\rangle = \frac{u_{q+e'}^\varepsilon - u_{q+e''}^\varepsilon}{\sqrt{3}} \langle v, e^\circ\rangle\langle v, e^*\rangle$$

Thus (2.8) preserves the properties of z^ε and u^ε listed above. Moreover, if

$$z_{q+e'}^\varepsilon - z_{q+e''}^\varepsilon = -\frac{u_{q+e'}^\varepsilon - u_{q+e''}^\varepsilon}{2} \frac{\Psi''(z_q^\varepsilon + u_q^\varepsilon/2) - \Psi''(z_q^\varepsilon - u_q^\varepsilon/2)}{\Psi''(z_q^\varepsilon + u_q^\varepsilon/2) + \Psi''(z_q^\varepsilon - u_q^\varepsilon/2)} \tag{2.11}$$

then $z_q^\varepsilon(t) \equiv z_q^\varepsilon$ and $u_q^\varepsilon(t) \equiv u_q^\varepsilon$ are stationary solutions to (2.8).

Corollary 2.3. Let $u_x\colon \mathbb{R} \mapsto \mathbb{R}$ be a bounded and continuously differentiable function, set $u_q^\varepsilon \equiv u_x(\varepsilon\langle q, e^*\rangle)\, e^\circ$, and define z_q^ε by (2.11), $z_q^\varepsilon = z_{q+e^\circ}^\varepsilon$ and $z_0^\varepsilon = z_{e'}^\varepsilon = z_0$ for all $q \in \mathbb{L}$ and $\varepsilon > 0$. If $\Gamma^\varepsilon(t) = (z_q^\varepsilon, u_q^\varepsilon)_{q\in\mathbb{L}}$ and the initial distributions μ^ε satisfy (2.4) for $\tau = 0$, then the conclusions of Theorem 2.2 hold true for all $\tau > 0$.

A microscopic model of the stationary flow of particles in a tube can be introduced by modifying the evolution law at the boundary. The associated stationary states and their hydrodynamic behavior are to be discussed in a forthcoming paper.

3. PROOF OF THEOREM 2.1

Our starting point is an entropy argument that goes back to Holley.[8] Since time is discrete and the Markov process defined by the modified FHP automata is not reversible, additional steps are needed.[5,2,9,4] We show that in a translation-invariant stationary state the specific entropy is constant, which implies in general that the state is reversible with respect to collisions and exchanges; thus we have to describe all reversible measures of $\mathbf{C}_\gamma$ and $\mathbf{B}_\beta$ separately. The case of collisions is trivial; the very concrete problem of $\mathbf{B}_\beta$ can be handled as follows.

For $\Lambda \subset \mathbb{L}$ let $\mathcal{F}_\Lambda$ denote the σ-field generated by the variables $\{\eta^X(q, v)\colon q \in \Lambda, v \in \mathbb{V}\}$. If $\Lambda = \Lambda^n$, then the notation $\mathcal{F}_n$ is used, and $\mathcal{F}_n(\mathcal{X})$ is the set of $\mathcal{F}_n$-measurable functions; $\mathbb{C}_0(\mathcal{X}) = \bigcup \mathcal{F}_n(\mathcal{X})$ is the set of local (cylinder) functions. Let $\mathcal{P}(\mathcal{X})$ denote the space of Borel probabilities on

$(\mathscr{X}, \mathscr{F})$, while $\mathscr{P}_0(\mathscr{X})$ is the set of translation-invariant elements of $\mathscr{P}(\mathscr{X})$. Especially within the text abbreviations like $\lambda[\phi]$ and $\lambda[\phi \mid \mathscr{F}_\Lambda]$ are used to denote expectation and conditional expectation. If $\mu, \lambda \in \mathscr{P}(\mathscr{X})$, then the relative entropy of μ with respect to λ is defined as

$$I[\mu \mid \lambda] = \sup \left\{ \int \phi \, d\mu - \log \int e^{\phi} \, d\lambda : \phi \in \mathbb{C}_0(\mathscr{X}) \right\} \tag{3.1}$$

The local relative entropy $I_n[\mu \mid \lambda]$ is obtained by restricting "sup" to $\phi \in \mathscr{F}_n(\mathscr{X})$; notice that $I_n[\mu \mid \lambda] = I[\mu_n \mid \lambda]$, where μ_n is an extension of μ from $\mathscr{F}_n$ defined by

$$\int \phi(X) \, \mu_n(dX) = \int \lambda[\phi \mid \mathscr{F}_n] \, d\mu \qquad \text{for} \quad \phi \in \mathbb{C}_0(\mathscr{X}) \tag{3.2}$$

In this section as a reference measure we are going to use the uniform Bernoulli measure $\lambda^0 \in \mathscr{G}_e$, i.e., $\lambda^0 = \lambda_\Gamma$ with $z_q = 0$ and $u_q = 0$ for all $q \in \mathbb{L}$. The specific entropy of $\mu \in \mathscr{P}_0(\mathscr{X})$ relative to λ^0 is then

$$\bar{I}[\mu \mid \lambda^0] \equiv \limsup_{n \to +\infty} \frac{I_n[\mu \mid \lambda^0]}{|\Lambda^n|} = \lim_{n \to \infty} \frac{I_n[\mu \mid \lambda^0]}{|\Lambda^n|} \tag{3.3}$$

The existence of the limit follows by subadditivity of entropy. Since the dominating measure λ^0 is a stationary state of the evolution, entropy and specific entropy are nonincreasing functions of time, and they are constant in a stationary regime. This relation yields some information on the reversible components **C** and **B** of the composed process. First we investigate this situation in a general framework.

Let $P = P[dY \mid X]$ denote a local and translation-invariant transition probability in $(\mathscr{X}, \mathscr{F})$, i.e., $P[\cdot \mid X] \in \mathscr{P}(\mathscr{X})$ for each $X \in \mathscr{X}$, $P[A \mid \cdot] \in \mathscr{F}_{n+1}(\mathscr{X})$ if $A \in \mathscr{F}_n$, and finally $P[\mathbf{s}^q A \mid \mathbf{s}^q X] = P[A \mid X]$ for all $X \in \mathscr{X}$, $q \in \mathbb{L}$, and $A \in \mathscr{F}$, where $\mathbf{s}^k$ denotes the shift by $k \in \mathbb{L}$, i.e., $\eta^{\mathbf{s}^k X}(q, v) \equiv \eta^X(q-k, v)$ for all $q \in \mathbb{L}$ and $v \in \mathbb{V}$; the shift of events, functions, and measures is defined accordingly. The associated operator of conditional expectation is $\mathbf{P}\phi = \mathbf{P}\phi(X) \equiv P[\phi \mid X]$, and a probability measure $\mu \in \mathscr{P}(\mathscr{X})$ is transformed by P to $\mu\mathbf{P}$ such that $\mu\mathbf{P}[\phi] = \mu[\mathbf{P}\phi]$. Assume that λ^0 is a stationary measure of P and let $\phi \in \mathbb{C}_0(\mathscr{X})$; then $\mathbf{P}\phi \in \mathbb{C}_0(\mathscr{X})$ and by convexity

$$\log \lambda^0[e^{\phi}] = \log \lambda^0[\mathbf{P}e^{\phi}] \geqslant \log \lambda^0[e^{\mathbf{P}\phi}]$$

Thus $I[\mu\mathbf{P} \mid \lambda^0] \leqslant I[\mu \mid \lambda^0]$ for any $\mu \in \mathscr{P}(\mathscr{X})$. Therefore if $\phi \in \mathscr{F}_n(\mathscr{X})$ and μ_n denotes the extension of $\mu \in \mathscr{P}(\mathscr{X})$ from $\mathscr{F}_n$ via (3.2), then by locality of P

$$\mu\mathbf{P}[\phi] = \mu_{n+1}\mathbf{P}[\phi] = \mu_{n+1}[\mathbf{P}\phi]$$

Consequently

$$I_n[\mu \mathbf{P} \mid \lambda^0] \leqslant I[\mu_{n+1} \mathbf{P} \mid \lambda^0] \leqslant I[\mu_{n+1} \mid \lambda^0] = I_{n+1}[\mu \mid \lambda^0]$$

whence for any $\mu \in \mathscr{P}_0(\mathscr{X})$ we have

$$\bar{I}[\mu \mathbf{P} \mid \lambda^0] \leqslant \liminf_{n \to \infty} \frac{I[\mu_n \mathbf{P} \mid \lambda^0]}{|\Lambda^n|} \leqslant \bar{I}[\mu \mid \lambda^0] \tag{3.4}$$

On the other hand, if $\varphi: \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$ depends only on a finite number of coordinates, then by convexity and (3.1) we get

$$\iint \varphi(X, Y)\, P[dY \mid X]\, \mu_n(dX) - \log \iint e^{\varphi(X, Y)} P[dY \mid X]\, \lambda^0(dX) \leqslant I[\mu_n \mid \lambda^0] \tag{3.5}$$

Thus choosing φ in a clever way, we obtain a lower bound for the decrement of $\bar{I}$.

Indeed, denote $P^* = P^*[dX \mid Y]$ the backward transition probability of the stationary process with initial distribution λ^0 and transition probability P, i.e., $P[dY \mid X]\, \lambda^0(dX) = P^*[dX \mid Y]\, \lambda^0(dY)$, and set

$$\varphi(X, Y) = \phi(X, Y) + \log f_n(Y) - \log \int e^{\phi(X, Y)} P^*[dX \mid Y]$$

where f_n is the density of $\mu_n \mathbf{P}$ with respect to λ^0; ϕ will be specified a bit later. From (3.5) by a direct calculation it follows that

$$\iint \phi(X, Y)\, P[dY \mid X]\, \mu_n(dX) - \int \log P^*[e^{\phi} \mid X]\, \mu_n \mathbf{P}(dY) \leqslant I[\mu_n \mid \lambda^0] - I[\mu_n \mathbf{P} \mid \lambda^0] \tag{3.6}$$

for any $\phi: \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$ depending only on a finite number of variables.

Suppose now that P^* is local in the following sense. P^* maps $\mathscr{F}_n(\mathscr{X})$ into $\mathscr{F}_{n+1}(\mathscr{X})$ and we have some $R > 0$ such that

$$\int \psi_1(X)\, \psi_2(X)\, P^*[dX \mid Y] = \int \psi_1(X)\, P^*[dX \mid Y] \int \psi_2(X)\, P^*[dX \mid Y] \tag{3.7}$$

whenever ψ_1 and ψ_2 are measurable with respect to $\mathscr{F}_{\Lambda_1}$ and $\mathscr{F}_{\Lambda_2}$, respectively, and the distance of the sets $\Lambda_1, \Lambda_2 \subset \mathbb{L}$ exceeds R. In the case of collisions and random exchanges, $P^* = P$, that is, λ^0 is a stationary and reversible measure of the transition probability P. The factorization property (3.7) also holds true with $R > 2$.

Lemma 3.1. Suppose that P is a transition probability satisfying the conditions listed above. If $\mu \in \mathscr{P}_0(\mathscr{X})$ and $\bar{I}[\mu \mathbf{P} \mid \lambda^0] = \bar{I}[\mu \mid \lambda^0]$, then

$$\iint \psi(X, Y)\, P[dY \mid X]\, \mu(dX) = \iiint \psi(X, Y)\, P^*[dX \mid Y]\, P[dY \mid Z]\, \mu(dZ) \tag{3.8}$$

for each $\psi: \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$ depending only on a finite number of variables.

Proof. Let $\psi_k(X, Y) \equiv \psi(\mathbf{s}^k X, \mathbf{s}^k Y)$ and choose ϕ of (3.6) as

$$\phi(X, Y) = \sum_{k \in \Lambda^n \cap m\mathbb{L}} \psi_k(X, Y)$$

where $m\mathbb{L} \equiv \{mq: q \in \mathbb{L}\}$ and $m \in \mathbb{N}$ is so large that the factorization property (3.7) applies. Since we have an $l \in \mathbb{N}$ such that $P^*[\psi_k \mid Y]$ is measurable with respect to $\mathscr{F}_n$ if $k \in \Lambda^{n-l}$, dividing both sides of (3.6) by $|\Lambda^n|$, using $\mu_n[\varphi] = \mu[\varphi]$ for $\varphi \in \mathscr{F}(\mathscr{X})$, and the translation invariance of μ, we get

$$\iint \psi(X, Y)\, P[dY \mid X]\, \mu(dX) \leqslant \int \log P^*[e^{\psi} \mid Y]\, \mu \mathbf{P}(dY)$$

and this inequality holds true for any multiple $\alpha\psi$ of ψ, too. Dividing by $\alpha \neq 0$ and letting $\alpha \to \pm 0$, we obtain the statement of Lemma 3.1. QED

Each step of the modified FHP automaton is composed of three transition rules P_S, P_C, and P_B. The free streaming P_S is completely deterministic; we write $\mathbf{S}\phi(X) \equiv \phi(\mathbf{S}X)$. The operators of conditional expectation $\mathbf{C}$ and $\mathbf{B}$ for collisions and exchanges are defined for every continuous $\phi: \mathscr{X} \mapsto \mathbb{R}$ by

$$\mathbf{C}\phi = \mathbf{C}\phi(X) \equiv P_C[\phi \mid X] \equiv \int \phi(\mathbf{C}_\gamma X)\, P(d\gamma)$$

$$\mathbf{B}\phi = \mathbf{B}\phi(X) \equiv P_B[\phi \mid X] \equiv \int \phi(\mathbf{B}_\beta X)\, P(d\beta)$$

The corresponding evolved measures are denoted as $\mu\mathbf{S}$, $\mu\mathbf{C}$, and $\mu\mathbf{B}$, respectively. Any one of the operators above maps $\mathscr{F}_n(\mathscr{X})$ into $\mathscr{F}_{n+1}(\mathscr{X})$ and preserves our referece measure λ^0; therefore $\bar{I}[\mu \mid \lambda^0] = \bar{I}[\mu\mathbf{C} \mid \lambda^0] = \bar{I}[\mu\mathbf{CB} \mid \lambda^0]$ whenever μ is a translation-invariant stationary state of the composed evolution, i.e., $\mu = \mu\mathbf{CBS}$. This relation does not imply that $\mu = \mu\mathbf{C}$ or $\mu\mathbf{CB} = \mu$. For example, $\bar{I}[\mu\mathbf{S} \mid \lambda^0] = \bar{I}[\mu \mid \lambda^0]$ for any translation-invariant μ, while $\mathbf{S}^* = \mathbf{S}^{-1}$; thus Lemma 3.1 does not say anything about

the free motion. On the other hand, $\mathbf{C}^* = \mathbf{C}$ and $\mathbf{B}^* = \mathbf{B}$; thus we have reversibility (detailed balance) of the randomized transitions. As a consequence we obtain that μ is symmetric with respect to the elementary transformations of the random mechanism separately.

For $q \in \mathbb{L}$ we define the maps $D'_q, D''_q: \mathscr{X} \mapsto \mathscr{X}$ by $D'_q X = D''_q X = X$ if v_q^X is not a dipole, and if $v_q^X = \mathbb{D}_v$, then D'_q and D''_q turn it into $\mathbb{D}_{v'}$ and $\mathbb{D}_{v''}$, respectively; other parts of the configurations are not changed. Rotation of a tripole at site q is executed by $T_q: \mathscr{X} \mapsto \mathscr{X}$ such that $T_q X = X$ if v_q^X is not a tripole and $v_q^X \to -v_q^X$ is the only change if $v_q^X = \mathbb{T}_v$ for some $v \in \mathbb{V}$. An elementary exchange $E_b^v: \mathscr{X} \mapsto \mathscr{X}$ for $b \in \mathbb{L}^*$, $v \in \mathbb{V}$ means that $\eta^X(k, v)$ and $\eta^X(j, v)$ are exchanged if $b = \{k, j\}$; the rest of the configuration is not altered.

Lemma 3.2. Suppose that $\mu \in \mathscr{P}_0(\mathscr{X})$ and $\bar{I}[\mu \mathbf{C} \mid \lambda^0] = \bar{I}[\mu \mid \lambda^0]$; then

$$\int \phi(D'_q X)\, \mu(dX) = \int \phi(D''_q X)\, \mu(dX) = \int \phi(T_q X)\, \mu(dX) = \int \phi(X)\, \mu(dX) \quad (3.9)$$

for all $q \in \mathbb{L}$ and $\phi \in \mathbb{C}_0(\mathscr{X})$.

Proof. Since $\mathbf{C}^* = \mathbf{C}$, choosing $\psi = \phi(X)$ in Lemma 3.1, we see that $\mu \mathbf{C}^2 = \mu$. A two-step transition $\mathbf{C}^2$ may change the configuration at a site $q \in \mathbb{L}$ only if v_q is a dipole or a tripole; then it is kept fix with probability 1/2, it is turned to right or to left with probability $1/4 - 1/4$ if v_q is a dipole, and it is reflected with probability 1/2 if v_q is a tripole; changes at different sites are independent of each other. This means that we essentially have a problem on finite-state Markov chains of dipole and tripole configurations. It is easy to check that their conditional distribution given the rest of the configuration does not depend on the condition; it is uniform Bernoulli, which proves Lemma 3.2. QED

Let us remark that $\mu \in \mathscr{P}_0(\mathscr{X})$ and $\bar{I}[\mu \mathbf{C}^o \mid \lambda^0] = \bar{I}[\mu \mid \lambda^0]$ imply only symmetry of μ with respect to D'_q and D''_q. The case of random exchanges is a little bit more complicated, but we get much more information on the distribution.

Lemma 3.3. Suppose that $\mu \in \mathscr{P}_0(\mathscr{X})$ and $\bar{I}[\mu \mathbf{B} \mid \lambda^0] = \bar{I}[\mu \mid \lambda^0]$; then

$$\int \phi(E_b^v X)\, \mu(dX) = \int \phi(X)\, \mu(dX) \quad (3.10)$$

for all $b \in \mathbb{L}^*$, $b \nparallel v \in \mathbb{V}$, and $\phi \in \mathbb{C}_0(\mathscr{X})$.

Proof. Let $X, Y \in \mathscr{X}$ and denote X_Λ the event that the random configuration coincides with X on $\Lambda \subset \mathbb{L}$. From Lemma 3.1 we know that

$$\mu[X_{\Lambda^{n+1}}] = \int_{Y_{\Lambda^n}} \frac{P_B[X_{\Lambda_{n+1}} | Z]}{P_B[Y_{\Lambda^n} | X]} \mu \mathbf{B}(dZ) \tag{3.11}$$

whenever $P_B[Y_{\Lambda^n} | X] > 0$. Since μ is translation invariant, it is enough to show that, e.g., if $b_1 = \{0, e^\circ\}$ and $v_1 \in \mathbb{V}$ is not parallel to b_1 then $\mu[E^{v_1}_{b_1} X_{\Lambda^{n+1}}] = \mu[X_{\Lambda^{n+1}}]$ for all $X \in \mathscr{X}$ and $n > 3$. To check this exchange symmetry of μ, we construct a distinguished configuration $Y = X^*$ depending on X in such a way that $Z \in X^*_{\Lambda^n}$ implies a Markov property

$$\frac{P_B[E^{v_1}_{b_1} X_{\Lambda^{n+1}} | Z]}{P_B[X_{\Lambda^{n+1}} | Z]} = \frac{P_B[E^{v_1}_{b_1} X^*_{\Lambda^n} | X]}{P_B[X^*_{\Lambda^n} | X]} \tag{3.12}$$

Indeed, as $P_B[E^{v_1}_{b_1} Y_{\Lambda^n} | Z] = P_B[Y_{\Lambda^n} | E^{v}_{b_1} X]$ is an identity, (3.11) and (3.12) imply the statement by a direct calculation. We may and do assume that $E^{v_1}_{b_1} X \neq X$.

To define X^*, let $\partial(X)$ denote the boundary of $X \in \mathscr{X}$; it is a set of marked bonds corresponding to effective exchanges:

$$\partial(X) \equiv \{(b, v): b \in \mathbb{L}^*, v \in \mathbb{V}, v \nparallel b \text{ and } E^v_b X \neq X\}$$

and consider a maximal set $\partial^*(X) \subset \partial(X)$ containing (b_1, v_1) such that if $(b, v), (c, u) \in \partial^*(X)$ are different, then $b \cap c = \varnothing$. Then

$$\omega^*(X) \equiv \{b \in \mathbb{L}^*: (b, v) \in \partial^*(X) \text{ for some } v \in \mathbb{V}\}$$

is a set of disjoint bonds.

Observe that $\partial^*(X)$ represents a maximal set of allowed changes of X, i.e., $(c, u) \in \partial(X)$ implies $b \cap c \neq \varnothing$ for some $(b, v) \in \partial^*(X)$. Now we define X^* as the configuration obtained from X by executing all possible exchanges prescribed by $\partial^*(X)$, that is,

$$X^* \equiv \prod_{(b,v) \in \partial^*(X)} E^v_b X$$

The dependence structure of the random exchange mechanism is represented by a set of binary variables $\chi = (\chi_b)_{b \in \mathbb{L}^*}$ such that $\chi_b = 1$ if $\beta_b \nparallel b$ and $\beta_c \parallel c$ for all adjacent bonds c, i.e., $|c \cap b| = 1$, and $\chi_b = 0$ otherwise. Indeed, an exchange across $b \in \mathbb{L}^*$ takes place if $\chi_b = 1$ but $\chi_c = 0$ for $|c \cap b| = 1$, and the way of exchange does not depend on the rest of the configuration of β.

To understand (3.12), let A^b be the event that $\omega(\beta) \ni b$, i.e., $\chi_b = 1$ but $\chi_c = 0$ whenever $|c \cap b| = 1$; denote $\bar{A}^b$ the complement of A^b, and set

$$A_n^X \equiv \prod_{\substack{b \subset \Lambda^n \\ b_1 \neq b \in \omega^*(X)}} A^b$$

In view of the construction of A_n^X, for any event $C \subset A_n^X$ depending on the values of χ_b for $b \not\subset \Lambda^{n-1}$ only, we have

$$P[A^{b_1} | A_n^X C] = P[A^{b_1} | A_n] = 4\kappa(1-2\kappa)^{10-m}$$

where $4\kappa = P[\beta_b \nparallel b]$ and m denotes the number of bonds c such that $|c \cap b_1| = |c \cap b| = 1$ for some $b \in \omega^*(X)$. This means that

$$\frac{P[\bar{A}^{b_1} A_n^X C]}{P[A^{b_1} A_n^X C]} = \frac{P[\bar{A}^{b_1} | A_n^X]}{P[A^{b_1} | A_n^X]} = c_n(X) \tag{3.13}$$

does not depend on C; therefore each side of (3.12) equals $c_n(X)/4$. Indeed, we have

$$\frac{P_B[E_{b_1}^{v_1} X_{\Lambda^n}^* | X]}{P_B[X_{\Lambda^n}^* | X]} = \frac{P[\bar{A}^{b_1} A_n^X C_n^X]}{4P[A^{b_1} A_n^X C_n^X]}$$

$$\frac{P_B[E_{b_1}^{v_1} X_{\Lambda^{n+1}} | Z]}{P_B[X_{\Lambda^{n+1}} | Z]} = \frac{P[\bar{A}^{b_1} A_n^X C_n^{X,Z}]}{4P[A^{b_1} A_n^X C_n^{X,Z}]}$$

where $C = C_n^X$ and $C = C_n^{X,Z}$ specify the bond structure of the prescribed transitions around the boundary of Λ^n from X and Z, respectively, which proves Lemma 2.2. QED

Now we are in a position to complete the proof of Theorem 2.1. Observe first that by means of a finite number of consecutive operations E_b^v we can exchange $\eta^X(q, v)$ and $\eta^X(p, v)$ for any pair $p, q \in \mathbb{L}$ and $v \in \mathbb{V}$ in such a way that the rest of the configuration is not altered. Therefore if $\mu \in \mathscr{P}_0(\mathscr{X})$ satisfies $\bar{I}[\mu \mathbf{B} | \lambda^0] = \bar{I}[\mu | \lambda^0]$, and $\phi \in \mathbb{C}_0(\mathscr{X})$ does not depend on the value of $\eta^X(q, v)$, then by Lemma 3.3

$$\int \eta^X(q, v)\, \phi(X)\, \mu(dX) = \int \eta^X(q, v)\, \phi(\mathbf{s}^k X)\, \mu(dX)$$

whenever $k \in \mathbb{L}$ is so far from q that neither does $\mathbf{s}^k \phi$ depend on $\eta^X(q, v)$, and thus for any finite set $\Lambda \subset \mathbb{L}$ consisting of such sites k we have

$$\int \eta^X(q, v)\, \phi(X)\, \mu(dX) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \int \eta^X(q, v)\, \phi(\mathbf{s}^k X)\, \mu(dX) \tag{3.14}$$

Sending $\Lambda \to \mathbb{L}$ in a suitable way, by the ergodic theorem [or by a weak $\mathbb{L}^2(d\mu)$ argument] we obtain that

$$\int \eta^X(q, v)\, \phi(X)\, \mu(dX) = \int \mu[\eta^X(q, v) \,|\, \mathscr{F}_{\text{inv}}]\, \mu[\phi \,|\, \mathscr{F}_{\text{inv}}]\, \mu(dX) \quad (3.15)$$

which means that almost every ergodic component $\mu[dX \,|\, \mathscr{F}_{\text{inv}}]$ of μ is a product measure.

Suppose now that $\mu \in \mathscr{P}_0(\mathscr{X})$ is a stationary state of the composed evolution; then

$$\bar{I}[\mu \,|\, \lambda^0] = \bar{I}[\mu \mathbf{C} \,|\, \lambda^0] = \bar{I}[\mu \mathbf{CB} \,|\, \lambda^0] = \bar{I}[\mu \mathbf{CBS} \,|\, \lambda^0]$$

whence $\mu = \mu \mathbf{C}$ by Lemma 3.2, while $\mu = \mu \mathbf{B}$ by Lemma 3.3. We know also that μ is a superposition of translation-invariant product measures, consequently $\mu = \mu \mathbf{S}$, which proves the first statement of Theorem 2.1.

Finally, if $0 < \rho_\infty^v \equiv \mu[\eta^X(0, v) \,|\, \mathscr{F}_{\text{inv}}] < 1$ μ-a.s. for each $v \in \mathbb{V}$, then $\mu[dX \,|\, \mathscr{F}_{\text{inv}}] = \lambda^w$ μ-a.s., where λ^w is a homogeneous product measure for $w = (w^v)_{v \in \mathbb{V}}$,

$$\lambda^w[X_\Lambda] = \prod_{q \in \Lambda} \prod_{v \in \mathbb{V}} \exp[w^v \eta^X(q, v) - \Psi(w^v)] \quad (3.16)$$

and w^v is specified by $\rho_v^\infty = \Psi'(w^v)$. From Lemma 3.2 we obtain immediately that each ergodic component λ^w is also symmetric with respect to the individual collision operators D_q', D_q'', and T_q; consequently

$$\begin{aligned} w^v + w^{-v} &= w^{v'} + w^{-v'} = w^{v''} + w^{-v''} \\ w^v + w^{v'} + w^{v''} &= w^{-v} + w^{-v'} + w^{-v''} \end{aligned} \quad (3.17)$$

This means that we have some uniquely determined $z \in \mathbb{R}$ and $u \in \mathbb{R}^2$ such that $w^v = z + \langle u, v \rangle$, namely

$$z = \tfrac{1}{12} \sum_{v \in \mathbb{V}} (w^v + w^{-v}), \qquad u = \tfrac{1}{6} \sum_{v \in \mathbb{V}} (w^v - w^{-v})\, v$$

which completes the proof. QED

Remark 3.4. If all tripoles are flipped simultaneously, i.e., if we choose the original collision mechanism $\mathbf{C}^o$, then we get $w^v = z + \langle u, v \rangle + \delta$ if $v \in \mathbb{V}^+ \equiv \{e^o, e', e''\}$ and $w^v = z + \langle u, v \rangle - \delta$ otherwise, where

$$\delta = \tfrac{1}{6} \sum_{v \in \mathbb{V}^+} (w^v - w^{-v})$$

The corresponding ergodic components λ^w of μ are preserved by **S**, **B**, and by the randomized rotation of dipoles, but the synchronized flipping of tripoles changes the sign of δ. The new parameter δ is associated with the quantity N^*,

$$N_q^*(X) \equiv \sum_{v \in \mathbb{V}^+} (\eta^X(q, v) - \eta^X(q, -v))$$

and if $\tilde{\mu}$ denotes the joint distribution of the parameters, then

$$\iiint e^{3z+3\delta} \tilde{\mu}(dz, du, d\delta) = \iiint e^{3z-3\delta} \tilde{\mu}(dz, du, d\delta)$$

is the criterion of stationarity. Since N^* is not conserved by $\mathbf{C}^o$, the derivation of the macroscopic equations seems to be problematic in that case.

4. PROOF OF THEOREM 2.2 AND ITS COROLLARY

Let $\lambda_t \equiv \lambda_{I^{e(t)}}$ be the time-dependent local equilibrium distribution we want to fit to the true state μ_t; the intermediate steps of the evolution are denoted as $\mu_t' \equiv \mu_t \mathbf{C}$ and $\mu_t'' \equiv \mu_t' \mathbf{B}$; thus $\mu_{t+1} = \mu_t'' \mathbf{S}$. Remember that most objects here and below depend also on the scaling parameter $\varepsilon > 0$. Let I_n^S denote the relative entropy of the joint distribution of the variables $\{\eta^X(q, v)\colon q + v \in \Lambda^n\}$ and let $g_n^s(t, X)$ be their joint distribution under λ_t; then

$$I_n[\mu_{t+1} | \lambda_{t+1}] = I_n^S[\mu_t'' | \lambda_{t+1} \mathbf{S}^{-1}] = I_n^S[\mu_t'' | \lambda_t] + D_n^S(t) \tag{4.1}$$

where, using the abbreviation $w_q^v(t) \equiv z_q(t) + \langle u_q(t), v \rangle$,

$$\begin{aligned} D_n^S(t) &\equiv \int \log \frac{g_n^s(t, X)}{g_n^s(t+1, \mathbf{S}X)} \mu_t''(dX) \\ &= \sum_{q+v \in \Lambda^n} w_q^v(t) \, \mu_t''[\eta(q, v)] - \Psi(w_q^v(t)) \\ &\quad - \sum_{q+v \in \Lambda^n} w_{q+v}^v(t+1) \, \mu_t''[\eta(q, v)] - \Psi(w_{q+v}^v(t+1)) \\ &\leqslant K_1 n^2 \varepsilon^2 + \sum_{q+v \in \Lambda^n} [w_{q+v}^v(t+1) - w_q^v(t)](\lambda_t[\eta(q, v)] - \mu_t''[\eta(q, v)]) \end{aligned} \tag{4.2}$$

as $\Psi'(w_q^v(t)) = \lambda_t[\eta(q, v)]$ and Ψ'' is bounded [cf. (2.9)], whence

$$D_n^S(t) \leqslant K_1 n^2 \varepsilon^2 + K_2 n^2 \varepsilon b(\varepsilon, \tau)$$
$$+ \sum_{q+v \in \Lambda^n} [w_q^v(t+1) - w_q^v(t)](\lambda_t[\eta(q, v)] - \mu_t''[\eta(q, v)])$$
$$+ \sum_{q+v \in \Lambda^n} [w_{q+v}^v(t) - w_q^v(t)](\lambda_t[\eta(q, v)] - \mu_t''[\eta(q, v)]) \quad (4.3)$$

Here and later on, K, K_1, K_2, K_2', etc., denote constants depending only on $a(\tau)$.

Now we can bring in the basic parameters z_q, u_q, their gradients (2.7), and the conservative quantities N_q, P_q as well. Again by smoothness we get

$$D_n^S(t) = K_1 n^2 \varepsilon^2 + K_2' n^2 \varepsilon b(\varepsilon, \tau) + K_3 n\varepsilon$$
$$+ \sum_{q \in \Lambda^{n+1}} \sum_{v \in \mathbb{V}} [z_q(t+1) - z_q(t)](\lambda_t[N_q] - \mu_t''[N_q])$$
$$+ \sum_{q \in \Lambda^{n+1}} \sum_{v \in \mathbb{V}} \langle u_q(t+1) - u_q(t), \lambda_t[P_q] - \mu_t''[P_q] \rangle$$
$$+ \sum_{q \in \Lambda^{n+1}} \sum_{v \in \mathbb{V}} \langle \nabla z_q(t), \lambda_t[P_q] - \mu_t''[P_q] \rangle$$
$$+ \sum_{q \in \Lambda^{n+1}} \sum_{v \in \mathbb{V}} \langle \nabla u_q(t)\, v, v \rangle (\lambda_t[\eta(q, v)] - \mu_t''[\eta(q, v)]) \quad (4.4)$$

The third correction $O(n\varepsilon)$ above is due to certain missing terms at the boundary of Λ^n. The critical sum is certainly the last one; observe that since $\langle \nabla uv, v \rangle = \operatorname{Tr} \nabla u Q_v$, it is just the trace of the sum of the tensors $G_q(t)$ for $q \in \Lambda^{n+1}$,

$$G_q(t) \equiv (\nabla u_q(t))\, Q(\rho_q(t), \pi_q(t)) - (\nabla u_q(t)) \int \sum_{v \in \mathbb{V}} \eta^X(q, v)\, Q_v\, \mu_t''(dX) \quad (4.5)$$

where $\rho_q(t) \equiv \lambda_t[N_q]$ and $\pi_q(t) \equiv \lambda_t[P_q]$. Suppose for a while that the following approximations are justified:

$$\sum_{v \in \mathbb{V}} \eta(q, v)\, Q_v \approx Q(N_q, P_q) \approx Q(\rho_q, \pi_q) + Q_\rho'(\rho_q, \pi_q)(N_q - \rho_q)$$
$$+ \langle Q_\pi'(\rho_q, \pi_q), P_q - \pi_q \rangle$$

where $\langle Q_\pi', P_q - \pi_q \rangle$ is a tensor whose matrix elements are scalar products,

$$\langle Q_\pi', P_q - \pi_q \rangle = \sum_{v \in \mathbb{V}} \langle \Phi_{v\pi}'(\rho_q, \pi_q), P_q - \pi_q \rangle\, Q_v$$

and $\Phi'_{v\pi}$ is the (vector-valued) derivative of Φ_v with respect to π. Then all terms of (4.4) cancel in view of the construction (2.8) of the parameters Γ, which is essentially what we have to show. To make the argument correct, let us introduce block averages Y^l_q for sequences $Y=(Y_q)_{q\in\mathbb{L}}$ by

$$Y^l_q \equiv \frac{1}{|\Lambda^l|}\sum_{k\in \mathbf{s}^q\Lambda^l} Y_k$$

In particular, the symbols $N^l_q(X)$ and $P^l_q(X)$ are defined in this way, and the block averages of the sequence

$$\zeta_q(X) \equiv \sum_{v\in\mathbb{V}} \eta^X(q,v)\, Q_v$$

will be denoted as $\zeta^l_q(X)$. A crucial step of the argument is the so-called one-block estimate of ref. 6 expressing the required ergodicity of the underlying process.

Lemma 4.1. We have

$$\lim_{l\to+\infty} \limsup_{\substack{T\to+\infty\\ n\to+\infty}} \frac{1}{n^2T}\sum_{t=0}^{T}\sum_{q\in\Lambda^n}\int R^l_q(X)\,\mu''_t(dX)=0 \tag{4.6}$$

where $R^l_q(X)\equiv\|\zeta^l_q(X)-Q(N^l_q(X),P^l_q(X))\|$.

Proof. The one-block estimate is a statement on the space-time averaged distribution

$$\bar{\mu}''^{n}_T \equiv \frac{1}{T+1}\frac{1}{|\Lambda^n|}\sum_{t=0}^{T}\sum_{q\in\Lambda^n}\mathbf{s}^q\mu''_t$$

Its weak limit points are translation-invariant stationary states of the composed evolution, thus by Theorem 2.1 they are identified as superpositions of equilibrium states from $\mathscr{G}_e$, and the proof is completed by the law of large numbers. QED

In view of the smoothness of parameters, we can replace $\zeta_q(X)$ by its block average $\zeta^l_q(X)$; the error of this step is only $n^2lo(\varepsilon)+nl^2O(\varepsilon)$; the second error term comes from the boundary of Λ^n. On the other hand,

$$\begin{aligned} Q(N^l_q,P^l_q) &= Q(\rho_q,\pi_q)+Q'_\rho(\rho_q,\pi_q)(N^l_q-\rho_q)\\ &\quad+\langle Q'_\pi(\rho_q,\pi_q),P^l_q-\pi_q\rangle+\tfrac{1}{2}Q''(\theta^l_q)(N^l_q-\rho_q,P^l_q-\pi_q) \end{aligned}\tag{4.7}$$

where $Q''(\theta)(\cdot, \cdot)$ is a quadratic form for each value of the intermediate point $\theta = \theta_q^l$. Comparing the above results, we obtain the following estimate.

Lemma 4.2. For $t\varepsilon \leqslant \tau < \tau_0$ we have a constant $K = K(\tau)$ such that

$$I_n[\mu_{t+1} | \lambda_{t+1}] \leqslant (1 + \varepsilon K)\, I_{n+1}[\mu_t'' | \lambda_t] + r_n^{\varepsilon, l}(t) + K\varepsilon \sum_{q \in \Lambda^{n-l}} \int R_q^l(X)\, \mu_t''(dX) \tag{4.8}$$

where $r_n^{\varepsilon, l}(t) = \varepsilon K n^2(\varepsilon + b(\varepsilon, \tau) + l/n + 1/l)$.

Proof. By convexity, $I_n^S[\mu_t'' | \lambda_t] \leqslant I_{n+1}[\mu_t'' | \lambda_t]$; thus we have to treat $D_n^S(t)$, which in view of Lemma 4.1 and (4.7) amounts to estimating the quadratic form Q''. Let I_q^l denote the relative entropy of the variables $\{\eta^X(p, v)\colon p - q \in \Lambda^l\}$; we prove

$$\int |Q''(\theta_q^l)(N_q^l(X) - \rho_q(t), P_q^l(X) - \pi_q(t))|\, \mu_t''(dX) \leqslant \frac{K_4}{|\Lambda^l|}(1 + I_q^l[\mu_t'' | \lambda_t] + \varepsilon^2 l^2) \tag{4.9}$$

From the variational definition (3.1) of entropy

$$\int [N_q^l(X) - \rho_q^l(t)]^2\, \mu_t''(dX) \leqslant \frac{1}{\alpha} I_q^l[\mu_t'' | \lambda_t] + \frac{1}{\alpha} \int \exp\{\alpha[N_q^l(X) - \rho_q^l(t)]\}\, \lambda_t(dX) \tag{4.10}$$

for all $\alpha > 0$, and a similar relation holds true also for the components of P_q^l. Choosing $\alpha = \alpha_0 |\Lambda^l|$, where α_0 is smaller than the large-deviation rate of N_q^l, we get a constant bound for the second term on the right-hand side of (4.10) directly from Cramer's theorem. In view of (2.9) both this constant and α_0 depend only on $a(\tau)$. Therefore if the matrix of the quadratic form $Q''(\theta_q^l)$ admits a fixed bound depending on $a(\tau)$, then we get (4.10) from (2.9).

The opposite event is possible only if θ_q^l is close to the boundary of the set of possible values of (ρ, π), which means a large deviation from the set of mean values prescribed by the uniform bound (2.9); thus (4.9) follows again from Cramer's theorem via (2.5). Indeed, from (4.7) we see that the value of Q'' is uniformly bounded even if we do not have any bound for the form $Q''(\theta_q^l)$, which completes the proof. QED

The next step is to estimate $I_n[\mu_t'' \mid \lambda_t]$; by convexity

$$I_n[\mu_t'' \mid \lambda_t] \leqslant \int I_{n+1}[\mu_t' \mid \lambda_t \mathbf{B}_\beta]\, P(d\beta) \leqslant I_{n+1}[\mu_t \mid \lambda_t] + \int D_{n+1}^{\beta}(t)\, P(d\beta) \tag{4.11}$$

as the collisions preserve λ_t, where in view of (2.3)

$$D_{n+1}^{\beta}(t) \equiv \int \log \frac{g_{n+1}(t, X)}{g_{n+1}(t, \mathbf{B}_\beta X)}\, \mu_t'(dX)$$

$$= \sum_{q \in \Lambda^{n+1}} \sum_{v \in \mathbb{V}} \{[w_q^v(t) - \tilde{w}_q^v(t)]\, \mu_t'[\eta(q, v)] - \Psi(w_q^v(t)) + \Psi(\tilde{w}_q^v(t))\}$$

$$\tilde{w}_q^v(t) \equiv w_q^v(t) + \sum_{y \in \mathbb{V}} [\chi_{q-y, y}^{v}(\beta)\, w_{q-y}^{v}(t) - \chi_{q, y}^{v}(\beta)\, w_q^v(t)]$$

Since the mean of $\chi_{q, y}^{v}$ is the same constant $\kappa(1-2\kappa)^{10}$ if $v \nparallel y$ and it is zero otherwise, expanding Ψ as before, we get

$$\int D_{n+1}^{\beta}(t)\, P(d\beta)$$

$$\leqslant K_1' n^2 \varepsilon^2 + \kappa(1-2\kappa)^{10}$$

$$\times \sum_{q \in \Lambda^{n+1}} \sum_{v \in \mathbb{V}} \sum_{y \nparallel v} [w_q^v(t) - w_{q-y}^v(t)](\mu_t'[\eta(q, v)] - \lambda_t[\eta(q, v)])$$

$$\leqslant K_1' n^2 \varepsilon^2 + K_5 n^2 \varepsilon b(\varepsilon, \tau) \tag{4.12}$$

from the second condition of (2.9) because the sum for $y \nparallel v$ is a second difference.

The above results can be summarized as follows. We have a constant K^* and some remainder $R_n^{\varepsilon, l}(t)$ such that for all $n \in \mathbb{N}$

$$I_n[\mu_{t+1}^{\varepsilon} \mid \lambda_{t+1}^{\varepsilon}] \leqslant (1 + \varepsilon K^*)\, I_{n+2}[\mu_t^{\varepsilon} \mid \lambda_t^{\varepsilon}] + R_{n+2}^{\varepsilon, l}(t) \tag{4.13}$$

whenever $0 \leqslant \varepsilon t \leqslant \tau < \tau_0$, where K^* depends only on $a(\tau)$, $R_n^{\varepsilon, l}(t)$ is an increasing function of $n \in \mathbb{N}$; moreover, for all $0 < \tau < \tau_0$ and $d > 0$ we have

$$\lim_{l \to +\infty} \limsup_{\varepsilon \to 0} \varepsilon^2 \sum_{t=0}^{T-1} R_n^{\varepsilon, l}(t) = 0 \tag{4.14}$$

uniformly in T and n if $\varepsilon T \leqslant \tau$ and $\varepsilon n \leqslant d$. This inequality can directly be solved by iteration; we obtain

$$I_n[\mu_T^{\varepsilon} \mid \lambda_T^{\varepsilon}] \leqslant (1 + \varepsilon K^*)^T\, I_{n+2T}[\mu_0^{\varepsilon} \mid \lambda_0^{\varepsilon}]$$

$$+ \sum_{t=0}^{T-1} (1 + \varepsilon K^*)^{T-t-1}\, R_{n+2T-2t}^{\varepsilon, l}(t) \tag{4.15}$$

which implies (2.4) by (4.14) and completes the proof of Theorem 2.2. In view of (2.11), the Euler prescribed parameters are also stationary; thus the proof of Corollary 2.3 follows in the same way.

ACKNOWLEDGMENTS

I wish to express my thanks to Joel L. Lebowitz for stimulating discussions, and in particular for drawing my attention to the problem of stationary solutions with or without boundary conditions. This work was finished at the Institut des Hautes Etudes Scientifiques, where I enjoyed the kind hospitality of Prof. M. Berger. This work was supported in part by Hungarian Science Foundation grant 1902 and by the Institut des Hautes Etudes Scientifiques.

REFERENCES

1. U. Frish, B. Hasslacher, and Y. Pomeau, Lattice gas automata for Navier–Stokes equation, *Phys. Rev. Lett.* **56**:1505 (1986).
2. J. Fritz, On the stationary measures of anharmonic systems in the presence of a small thermal noise, *J. Stat. Phys.* **44**:25–47 (1986).
3. J. Fritz, Hydrodynamics of FHP automata via OYV and ergodicity, Notes, Rutgers University (1990).
4. J. Fritz, T. Funaki, and J. L. Lebowitz, Stationary states of random Hamiltonian systems, *Probab. Th. Rel. Fields* **99**: 211–236 (1994). Preprint, MKI Budapest (1992).
5. G. Gallavotti and M. Verboven, On the classical KMS condition, *Nuovo Cimento* **28B**:274–286 (1976).
6. M. Z. Guo, G. Papanicolau, and S. R. S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions, *Commun. Math. Phys.* **118**:31–53 (1988).
7. J. Hardy, O. dePazzis, and Y. Pomeau, Molecular dynamics of classical lattice gas: Transport properties and time correlation functions, *Phys. Rev. A* **13**:1949 (1976).
8. R. Holley, Free energy in a Markovian model of a lattice spin system, *Commun. Math. Phys.* **23**:87–99 (1971).
9. S. Olla and S. R. S. Varadhan, Scaling limit for interacting Ornstein–Uhlenbeck processes, *Commun. Math. Phys.* **135**:355–378 (1991).
10. S. Olla, S. R. S. Varadhan, and H. T. Yau, Derivation of Euler equations from Hamiltonian dynamics with weak noise, *Commun. Math. Phys.* **155**:523–560 (1993).
11. J. Smoller, *Shock Waves and Reaction-Diffusion Equations* (Springer-Verlag, New York, 1983).
12. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, New York, 1991).
13. H. T. Yau, Relative entropy and hydrodynamics of Ginzburg–Landau models, *Lett. Math. Phys.* **22**:63 (1991).